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Periodic Solutions of Certain Liénard Equations with Delay

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The nonlinear, autonomous differential equation

$$x + f(x)\,\dot{x} + g(x) = 0 \tag{A}$$

is referred to as a generalized Liénard equation. Conditions on $f(x)$ and $g(x)$ for which (A) will have periodic solutions are well known. See, for example, [3, 8, 9]. It is the purpose of this paper to show that an equation similar to (A) but with a particular history-dependent restoring force has a periodic solution. The equation is

$$x(t) + f(x(t))\,\dot{x}(t) + g(x(t-r)) = 0, \qquad r > 0. \tag{S}$$

In [7, Section 31] it is shown that (S) has a periodic solution with period greater than $2r$, when f is even and g is odd with continuous first derivative. The proof in [7] closely parallels that of the author [4] in which a special case of (S),

$$x(t) + \epsilon(x^2(t) - 1)\,\dot{x}(t) + x(t-r) = 0,$$

was shown to have a periodic solution for all $r > 0$ and $\epsilon > 0$. The above-mentioned conditions on f and g are not required in the following theorem.

THEOREM. *Equation* (S) *has a nontrivial periodic solution, period greater than* $2r$, *if*

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(I) *$f(x) = -a + f_1(x)$ is continuous for all x; $f_1(0) = 0$;*

$$F(x) = \int_0^x f(s)\, ds$$

is monotone for all large $|x|$; and $F(x) \to +\infty(-\infty)$ as $x \to +\infty(-\infty)$,

(II) *$g(x) = x[1 + g_1(x)]$, $g(x)$ is nondecreasing; $g_1(x)$ is continuous for all x; $g_1(0) = 0$; and $xg(x) > 0$ for all $x \neq 0$,*

(III) *$F^{-1}(y)\, g(F^{-1}(y))/y \to 0$ as $|y| \to \infty$,*

(IV) *if γ_0 is the solution of $\gamma^2 = \cos r\gamma$, $0 < \gamma < \pi/2r$, then $a\gamma_0 > -\sin r\gamma_0$.*

Conditions (I) and (II) are the same type as required of f and g in the study of (A). Usually f is assumed even. This simplifies proofs, but is not necessary. Condition (III) arises in the proof where a boundedness condition is required. Note that condition (III) is satisfied for a van der Pol type equation $f(x) = x^2 - 1$, $g(x) = x$. Condition (IV) is a necessary and sufficient condition that at least one eigenvalue of the linear approximation to (S) have positive real parts. This is a necessary aspect of the theory of periodic solutions of functional differential equations described in [4].

Proof of the Theorem

The proof of the theorem is an application of Theorem 2.1 of [4]. Before stating that theorem, certain background details and notation will be presented. More extensive discussion and details are in [4–7].

Let r be a fixed positive number. Denote by C the set of functions continuous on the closed interval $[-r, 0]$ with range in the space of complex n-vectors E^n; $C = C([-r, 0], E^n)$. The norm for C is the uniform norm denoted by $\|\cdot\|$. The boundary of a set G in C is denoted by ∂G. The closed ball of radius ρ about the origin $\{0\}$ of C is denoted by $B(\rho)$. A cone K in C is a closed convex set such that if $k \in K$ then $\beta k \in K$ for all $\beta \geqslant 0$, and at least one of g, $-g$ in C is not in K if $g \neq \{0\}$. Let x be a continuous function on $[-r, T)$, $T > 0$, and range in E^n. Then for each fixed t, $0 \leqslant t < T$, the symbol x_t denotes a function in C whose graph coincides with the restriction of x to the interval $[t - r, t]$. That is, $x_t(\theta) \equiv x(t + \theta)$ for $-r \leqslant \theta \leqslant 0$.

An autonomous linear differential equation is

$$\dot{x}(t) = L(x_t) = \int_{-r}^{0} [d\beta(\theta)]\, x(t + \theta), \tag{1}$$

where $L : C \to E^n$ is a continuous linear operator. The $n \times n$ matrix β has components which are of bounded variation on $[-r, 0]$. A solution of (1) with initial function φ is a continuous function $x(t)$ on $[-r, T)$, $T > 0$, such that $x(t)$ coincides with φ on $[-r, 0]$ and satisfies (1) on $[0, T)$ (right hand derivate at $t = 0$). The solution trajectory is the set $\{x_t\}$, $0 \leqslant t < T$.

The characteristic equation associated with (1) is

$$\det \Delta(\lambda) = 0, \qquad \Delta(\lambda) = I - \int_{-r}^{0} [d\beta(\theta)]\, e^{\lambda\theta}, \tag{2}$$

where I is the $n \times n$ identity matrix. The zeros of (2) are called the eigenvalues of (L). They are real or occur in complex conjugate pairs, and only a finite number (counting multiplicity) of them lie to the right of any given vertical line in the complex plane. Let $\Lambda = \{\lambda : \det \Delta(\lambda) = 0,\ \operatorname{Re} \lambda > 0\}$, be nonempty; then corresponding to Λ are coordinate subspaces, $P(\Lambda) = P$ and $Q(\Lambda) = Q$ of C. These are invariant manifolds for (1), since if φ is in P or Q, the solution trajectory remains in P or Q for all $t \geqslant 0$. Subspace P is distinguished by the fact that it is the unstable manifold of the origin. Also P has finite dimension, and Q has codimension equal to that of P. Any g in C has a unique representation $g = g^P + g^Q$, where g^P is in P and g^Q is in Q.

Theorem 2.1 of [4] concerns the equation

$$\dot{x}(t) = L(x_t) + N(x_t), \tag{3}$$

where L is continuous and linear, and at least one eigenvalue of (L) has positive real part. The operator $N : C \to E^n$ is continuous, not linear, and $N(\{0\}) = 0$. Given a $\delta > 0$, there is a continuous, nondecreasing function η, $\eta(0) = 0$ such that for any g, f in $B(\delta)$, $| N(g) - N(f)| \leqslant \eta(\delta) \| g - f \|$. The assumption on L ensures that Λ is nonempty, so that $P(\Lambda)$ exists. Suppose now that the trajectory of a solution (in C) of (3) starts from any element $k \neq \{0\}$ in a fixed cone K, leaves the cone, and returns to the cone at a time $\tau(k)$. Let $\tau(k)$ be bounded above for all uniformly bounded k and be continuous and bounded below by r. Denote the solution by $x(k)$, and define an operator A by

$$Ak \overset{\text{def}}{=} x(k)_{\tau(k)} . \tag{4}$$

If A takes bounded sets into bounded sets, A is a completely continuous operator. Theorem 2.1 of [4] then states the following:

Assume that the mapping operator A of (4) is completely continuous, maps a cone K in C into itself, and satisfies the conditions:

(i) If G is any open bounded neighborhood of $\{0\}$, then $\inf \| Ak \| > 0$ if $k \in \partial G \cap k$,

(ii) For some $N > 0$, and all $k \in K$ such that $\| k \| \geqslant N$, $\| Ak \| < \| k \|$.

Assume also that the cone K satisfies the condition

(iii) $\inf \| k^P \| > 0$, $k \in \partial B(1) \cap K$, $P = P(A)$.

Under these hypotheses there exists one nontrivial fixed point A in K, which corresponds to a nontrivial periodic solution of (3) with period greater than r. Condition (ii) may be replaced by the weaker condition;

(iia) If φ is an eigenfunction of A, $A\varphi = \mu\varphi$, then there is an N such that $\| \varphi \| > N$ implies $0 < \mu < 1$.

Also, the cone K of Theorem 2.1 of [4] may be replaced by a truncated cone with no change in the results.

The application of the above theorem to a specific equation such as (S) consists of three parts: First, A and K must be chosen so that $A : K \to K$. Second, A must be shown to satisfy conditions (i) and (ii). Third, K must be shown to satisfy condition (iii).

To establish the first part, we put (3) into the form

$$\dot{w}(t) = \begin{Bmatrix} \dot{x}(t) \\ y(t) \end{Bmatrix} = \begin{Bmatrix} y - F(x) \\ -g(x(t-r)) \end{Bmatrix}, \tag{5}$$

and prove certain characteristic properties of the solution. These permit us to establish K, define A, and prove that the properties (i) and (iia) hold.

The Banach space $C_0 = \{ g = (f, a)^T, f \in C([-r, 0], E^1), a \in E^1\}$ is used, since a solution of (5) is uniquely defined for all $t > 0$ if it starts from an element g of C_0 at $t = 0$. The trajectory of the solution is the set in C_0, $\{w(g)_t\}$, $t \geqslant 0$. We define the projection of the trajectory as the set of points $\{w(g)(t)\}$ in the (x, y) plane. In the following, we refer to this set as "the projection".

A satisfactory cone K for our purposes turns out to be the set of functions $K = (k_1(\phi), k_2)^T$ in C_0 such that $k_1(-r) = 0$, k_1 is not decreasing on $[-r, 0]$, and $k_2 \geqslant 0$. The following lemma states essential properties of the solution with respect to K.

LEMMA 1. *Define* $-K = \{-k, k \in K\}$. *A solution trajectory of* (5) *starting from any* $k \neq 0$ *in* K *leaves* K. *After a finite time greater than* r, *it enters* $-K$. *After another finite time greater than* r, *it reenters* K.

The projection of a typical trajectory is shown in Fig. 1. Corresponding to the points 0,..., 8 of the figure are the times t_i, $i = 0,..., 8$. That is, t_1 and t_5 are the times when the projection crosses the curve $y = F(x)$, t_2 and t_6

are times when it crosses the x axis, t_3 and t_7 are the times when it crosses the y axis, and $t_4 = t_3 + r$, $t_8 = t_7 + r$ are times when $\dot{y} = 0$ (i.e., when $g(x) = 0$). The times t_i depend on k.

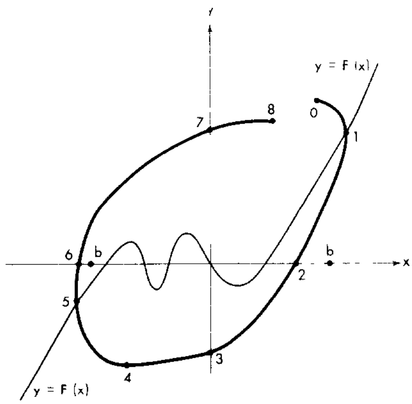

FIG. 1. Projection of a solution to (5).

Proof of Lemma 1. The proof of Lemma 1 is the same as the proof of Lemma 31.1 of [7] except for a slight modification to account for the possibility of f or g not being symmetrical.

The above lemma suggests that we define an operator on K as follows:

$$Ak = w(k)_{\tau(k)}\,, \qquad \tau(k) \overset{\text{def}}{=} t_7(k) + r, \qquad k \in K,$$

where t_7 is the first time at which $x(k) = 0$, $\dot{x}(k) > 0$ and $y(k) > 0$. Lemma 1 shows that $t_7(k)$ exists for all $k \neq \{0\}$ in K, and thus that $A : K \to K$ is a well-defined operator. Clearly, $\tau(k) > 2r$. That A takes bounded sets into bounded sets, as well as the continuity and boundedness requirements for $\tau(k)$ can be established by duplicating the proof of Lemma 31.1 of [7].

The second part of the proof of the theorem consists of showing that conditions (i) and (iia) hold. That $\inf \| Ak \| > 0$ for $k \in \partial G \cap K$ can be shown by the method of Lemma 29.3 of [7]. The method consists of assuming that a sequence of solutions $w(k_j)_{t_8} = Ak_j$ tend to 0 as $j \to \infty$, and then obtaining a contradiction

We show that (iia) rather than (ii) holds. In order to do this, we first prove some more detailed properties of the solution projection.

LEMMA 2. *Let $b > 0$ be such that $F(b) > 0$, $F(-b) < 0$ and $F^{-1}(y)$ is monotone for all $y \geqslant F(b)$ and $y \leqslant F(-b)$. Let $c = \max F(x)$, $0 \leqslant x \leqslant b$, and define $M = c + [2bg(b)]^{1/2}$. Then for any $k \in K$ with $k_2 \geqslant M$, we have*

$y(k)(t) \geqslant F(b)$ *for* $0 \leqslant t \leqslant t_1$, i.e., *the projection crosses the curve* $y = F(x)$ *before it crosses the* x *axis.*

Proof. If $k_1(0) = x_0 \geqslant b$, then clearly the result is true (Lemma 1). Suppose now that $0 \leqslant k_1(0) \leqslant b$. Then we have the following inequality for the slope of the projection when $0 \leqslant x(t) \leqslant b$, $y(t) > c$:

$$\frac{dy}{dx} = -\frac{g(x(t-r))}{y - F(x)} \geqslant -\frac{g(b)}{y-c}.$$

The associated differential equation

$$\frac{du}{dx} = -\frac{g(b)}{u-c}, \qquad u(0) = M$$

has solution $u(x) = c + [2g(b)(b-x)]^{1/2}$. Since the slope of u is a lower bound on dy/dx for $0 \leqslant x \leqslant b$, the projection cannot cross the curve $y = u(x)$ on $0 \leqslant x \leqslant b$, and $y(t) > c \geqslant F(b)$ for $0 \leqslant t \leqslant t_1$.

LEMMA 3. *Let* k *be in* K *with* $k_2 \geqslant M$. *Then*

$$\inf |\, y(k)(t) - F(x(k)(t))| \geqslant m > 0,$$

where the infimum may be taken over $t_2 \leqslant t \leqslant t_3$ *or* $t_6 \leqslant t \leqslant t_7$ (*when* $t_5 < t_6$).

Proof. Assume that the lemma is false for the case $t_2 \leqslant t \leqslant t_3$. Then there must be a sequence $\{k_j\} \in K$, $k_{1j}(0) \geqslant M$, and a corresponding sequence of times t_j, $t_2 \leqslant t_j \leqslant t_3$, $j = 9, 10, \ldots$, with the property that

$$\lim |\, y_j - F(x_j)| = 0, \qquad \text{as} \qquad j \to \infty. \tag{6}$$

Here $y_j = y(k_j)(t_j)$, etc. Also t_2 and t_3 depend on k_j, but there is no need to denote this. Equation (6) implies that the sequence of points $\{(x_j, y_j)\}$ converges to a point on the curve $y = F(x)$. By Lemma 1, the point must be on a portion of the curve in quadrant IV. Provided that $g(x_j(t_j - r)) \not\to 0$ as $j \to \infty$, the slope of the projection at (x_j, y_j),

$$\left(\frac{dy}{dx}\right)_j = -\frac{g(x_j(t_j - r))}{y_j - F(x_j)} \to \infty, \qquad \text{as} \quad j \to \infty.$$

This contradicts the fact that the slope of the curve $y = F(x)$ is bounded in quadrant IV. We must verify that $g(x_j(t_j - r))$ cannot tend to zero. Suppose it did. Then $x_j(t) \to 0$ for all t in $[-r, t_j - r]$, because if $x(t)$ becomes positive, it cannot become zero again until t_3.

We thus have

$$k_{1j}(0) - y(t_j) = \int_0^{t_j} g(x_j(s-r))\, ds \to 0,$$

which contradicts the fact that the left hand side is greater than M. The case $t_6 \leqslant t \leqslant t_7$ $(t_5 < t_6)$ is proven in a similar manner.

We now verify that (iia) holds. Let $P\phi = (x_0, y_0)$ be the projection of an eigenfunction ϕ.

First suppose that $P\phi$ lies under the curve $y = F(x)$ and to the right of $x = F^{-1}(M)$. Then $PA\varphi = P\mu\varphi = (x_8, y_8) = (\mu x_0, \mu y_0)$ lies on the same straight line through the origin as (x_0, y_0). By Lemma 1, the point (x_8, y_8) must lie above the curve $y = F(x)$ in such a manner that $x_8 < x_0$. Thus $0 < \mu < 1$.

Second suppose that $P\phi$ lies above the curve $y = F(x)$. Then we use the following estimates to show that $\mu y_0 = y_8 < y_0$. First, it is necessary to show $-y_5 > -y_0$ providing y_0 is sufficiently large. This follows from

$$t_3 - t_2 = \int_{t_2}^{t_3} dt = \int_{x_2}^{0} \frac{dx}{y - F(x)} \leqslant \frac{x_2}{m} < \frac{x_1}{m},$$

$$|y_5| = \int_{t_2}^{t_5} g(x(t-r))\, dt < \int_{t_2}^{t_4} g(x(t-r))\, dt < g(x_1)(t_3 - t_2 + r),$$

$$|y_5| < g(x_1)\left[\frac{x_1}{m} + r\right] < g(F^{-1}(y_0))\left[\frac{F^{-1}(y_0)}{m} + r\right]. \tag{7}$$

The monotonicity property of g has been used. By condition III, the right hand side of (7) will be less than y_0 for any y_0 sufficiently large.

We next show $y_8 < y_0$ for y_0 sufficiently large. Let $t^* = t_6$ when $t_5 \leqslant t_6$ and $t^* > t_5$ when $t_5 > t_6$. Also let

$$m^* = \inf |y(k)(t) - F(x(k)(t))|, \qquad t^* \leqslant t \leqslant t_7.$$

Then the estimates

$$t_7 - t^* = \int_{t^*}^{t_7} dt = \int_{x^*}^{0} \frac{dx}{y - F(x)} < \frac{-x^*}{m^*} < \frac{-x_5}{m^*}$$

and

$$y_8 = -\int_{t^*}^{t_8} g(x(t-r))\, dt < |g(x_5)|\,(t_7 - t^* + r)$$

imply

$$y_8 < |g(x_5)|\left[\frac{|x_5|}{m^*} + r\right]. \tag{8}$$

If F^{-1} is monotonic for $y < y_5$, then, since $-y_0 < y_5$, we have the bound

$$y_8 < |g(F^{-1}(-y_0))| \left[\frac{|F^{-1}(-y_0)|}{m^*} + r\right]. \tag{9}$$

This will be bounded by y_0 for all sufficiently large y_0. Suppose $F^{-1}(y)$ is not monotonic for $y < y_5$. Then $F^{-1}(-y_0)$ could have one or several values. By observing the graph of a typical $y = F(x)$ it is clear that at least one value of $F^{-1}(-y_0)$ must be less than x_5. Expressions (8) and (9) and the conclusion follow at once.

The final part of the proof consists in showing that the cone K satisfies property (iii): $\inf \| k^P \| > 0$, $k \in \partial B(1) \cap K$. This is shown to be true in Lemma 31.5 of [7], providing IV holds. Condition IV ensures that the characteristic Eq. (2), which in this case is $\lambda^2 - a\lambda + \exp(-r\lambda) = 0$, has a zero with positive real part. This in turn means that the subspace $P = P(\Lambda)$ is not void, so that k^P has meaning. This completes the proof of the theorem.

Concluding Remarks. There is no loss of generality by writing $g(x)$ as in II, since if $g(x) = x[b + g_1(x)]$, then the $b > 0$ can be eliminated by a change in the variable t. The theorem still holds if

$$g(x) = \begin{cases} x[b + g_1(x)] & x > 0 \\ x[c + g_2(x)] & x < 0 \end{cases},$$

where $b, c > 0$. If $b > c$, it suffices to take condition IV to be $a\gamma_1 > -c^2 \sin(r\gamma_1)$, where $\gamma_1{}^2 = c^2 \cos(r\gamma_1)$. Then let $\gamma_0{}^2 = b^2 \cos(r\gamma_0)$. The inequality $-b^2 \sin(r\gamma_1) < -c^2 \sin(r\gamma_0) < a\gamma_1 < a\gamma_0$ follows, so condition IV holds whether x is greater or less than zero.

It would be desirable to let the restoring force be more general. For example, $g(x_t)$ where $g(c)$, $c =$ constant function, enjoys properties similar to II and III. However, the calculations of the proof do not carry over even in the case $g(x_t) = bx(t) + x(t - r)$, $b > 0$. The difficulty is that, although the projection curve still has the aspect of that of the figure, the time $t_4 - t_3 < r$ varies with k rather than being a constant as before. In the proof we relied very heavily on the fact that $t_4 - t_3 = r$. This has prevented the author from finding a suitable cone for Theorem 2.1 of [4]. The most useful class of cones are those whose elements are nondecreasing over a fixed interval, e.g., $[-r, 0]$. No such cone seems satisfactory here. Another aspect of this problem is that a lower bound on the period of oscillation of a solution cannot be found. Numerical studies indicate, however, that equations of form (S) but with the g term $Ax(t) + Bx(t - r)$ have periodic solutions for a range of the A, B, r parameters.

Condition III appears artificial at first, but certainly seems quite necessary when the details of the proof are examined. This condition could be eliminated if some sort of boundedness property of solutions could be identified; for example, if for some M, $\| k \| \geqslant M$, there is a time $\tau = t_8$ so that the projection $\| w_\tau(k) \| \leqslant M$. This might possibly be done by devising an appropriate "energy" function for (S). Probably no new conditions on f and g would be necessary. Note that Condition III imposes the following restriction on f and g: suppose $g(x) = x + Ax^3$; then if $f(x) = x^2 - 1$, condition III is not satisfied. However, if $f(x)$ has an x^4 or higher term, condition III is satisfied. Numerical studies of the equation

$$x(t) + (x^2(t) - 1)\,\dot{x}(t) + x(t - r) + Ax^3(t - r) = 0$$

indicate that periodic solutions occur for small A. If A is too large, the computed solution grows, seemingly without bound. However, numerical studies of the same equation with the x^2 term replaced by x^4 indicate that periodic solutions occur for all values of A within the range of numerical stability.

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